Physics 511: Electrodynamics

Spring 2018

Midterm Exam #2

April 23, 2018

Instructions:

- Do any 2 of the 3 problems. All problems carry equal weight.
- This is a closed-book closed-note exam.
- You may use personal notes that fit on a doubles-sided $A4$ paper.

1- A plane wave of frequency ω is incident at angle θ from vacumm on the front surface of a plane-parallel dielectric non-magnetic slab of thickness d and index of refraction n (where n is real). The back surface of the slab is coated with a *perfectly* conducting film. Assume that the wave has s polarization.

(a) Write down the transfer matrix for the vacuum-dielectric interface as well as the propagation matrix for propagation in the dielectric slab out to its conducting back surface. Use the boundary conditions to relate the two traveling components of the electric field at the conducting interface.

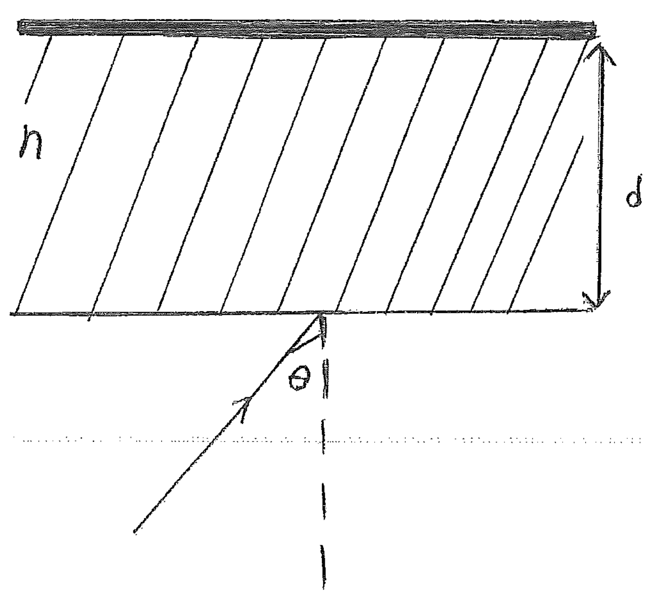
(b) Show that the amplitude reflection coefficient for the slab may be expressed as follows:

$$r = -\frac{r_s \exp(-i\alpha) + \exp(i\alpha)}{r_s \exp(i\alpha) + \exp(-i\alpha)},$$

where

$$r_s \equiv \frac{\sqrt{n^2 - \sin^2\theta} - \cos\theta}{\sqrt{n^2 - \sin^2\theta} + \cos\theta} \quad , \quad \alpha \equiv \frac{\omega d \sqrt{n^2 - \sin^2\theta}}{c}.$$

Show that $|r| = 1$ and interpret this result.

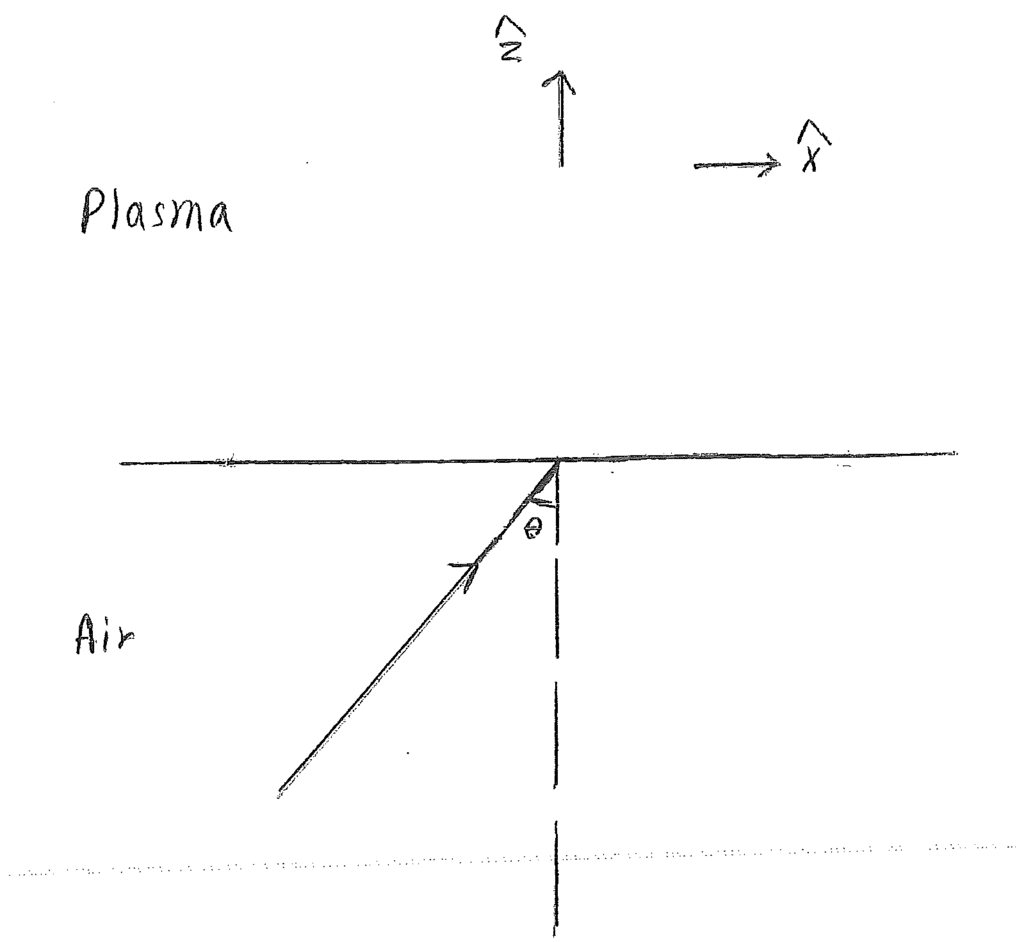


2- A monochromatic wave of frequency ω is traveling inside the air and is incident at angle θ at its planar boundary with a plasma that has plasma frequency ω_{p}.

(a) Find the minimum value of ω (as a function of ω_{p} and θ) that allows the refracted wave to propagatate along the z direction in the plasma.

(b) Repeat part (a) when a static magnetic field $\vec{B} = B_0\hat{x}$ is present inside the plasma (take $B_0 > 0$). Express your result up to linear terms in the cyclotron frequency ω_{B}.

(c) Repeat part (b), but find the lowest value of ω that allows a *linearly* polarized wave to propagate along the z direction in the plasma.



3- Consider the angular momentum of the electromagnetic field of a circularly polarized Bessel beam traveling in vacuum in the z direction. Such a beam is represented by the following $\vec{E}$ and $\vec{B}$ fields (in complex notation):

$$\vec{E} = \left[E_0\ \hat{e}_+ + \frac{i}{k}\left(\frac{\partial E_0}{\partial x} + i\frac{\partial E_0}{\partial y}\right)\hat{z}\right]\exp(ikz - i\omega t) \quad , \quad \vec{B} = -\frac{i}{c}\vec{E}\,,$$

where $\hat{e}_) = (\hat{x} + i\hat{y})/\sqrt{2}$ and $k \approx \omega/c$. The specific form of E_0 is best expressed in polar coordinates as

$$E_0(\rho, \phi) = A\ J_m(\gamma\rho)\ \exp(im\phi)\,,$$

where J_m is the Bessel function of the first kind of order m and $\gamma \ll k$.

(a) Show that the z component of the time-averaged total angular monetum may be expressed as follows:

$$\langle J_z\rangle = \frac{\epsilon_0}{2}\ \mathrm{Re}\int d^3x\ (\hat{z}\times\vec{x})\cdot(\vec{E}\times\vec{B}^*)\,.$$

(b) Show that the expression in part (a) may be written as

$$\langle J_z\rangle = \frac{\epsilon_0}{2ck}\int\left(m|E_0|^2 - \frac{\rho}{2}\frac{\partial|E_0|^2}{\partial\rho}\right)d^3x\,.$$

You may use the following result without proof:

$$\left(\frac{\partial E_0}{\partial x} + i\frac{\partial E_0}{\partial y}\right) = \exp(i\phi)\left(\frac{\partial E_0}{\partial\rho} + \frac{i}{\rho}\frac{\partial E_0}{\partial\phi}\right)\,.$$

(c) By performing an integration by parts over ρ, show that the result of part (b) may be expressed as

$$\langle J_z\rangle = \frac{\epsilon_0}{2ck}(m+1)\int|E_0|^2 d^3x\,.$$

Interpret this result in terms of photon angular momentum.